One Year ANR Post Doctoral Fellowship; Mathematics and Algorithms for Dynamic Cone-Beam CT and ROI Reconstruction A Mathematical Introduction

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Computed tomography aims at reconstructing images of internal physical quantities (attenuation coefficients, radioactivity concentrations) from external measurements (X-ray projections, radiation detectors). To first order, these measurements can be mathematically modeled by the Radon transform: straight line integrals of the unknown function. The problem to solve is the reconstruction of a function from a set of line integrals. Solving this problem led to the development in medicine of CT scanners, PET scanners and SPECT scanners.

Let $\vec{v} \in \mathbb{R}^n$ denote a point (an X-ray source position in X-ray CT $n = 2$ or CB CT $n = 3$), and $\vec{\zeta} \in \mathbb{S}^{n-1}$ a unit vector (in the direction from the source to the detector in X-ray CT), the Divergent Beam transform is defined by

$$\mathcal{D}\mu\left(\vec{v}, \vec{\zeta}\right) = \int_0^{+\infty} \mu\left(\vec{v} + l\vec{\zeta}\right) dl \tag{1}$$

Generally (in particular in X-ray CT, see Fig. 1 for the Fan Beam geometry) the data are acquired from multiple source positions and the source follows a trajectory along a curve

$$\begin{array}{rcl} \vec{v} \,:\, T \subset \mathbb{R} & \longrightarrow & \mathbb{R}^n \\ t & \longrightarrow & \vec{v}(t) \end{array}$$

We will suppose in the following that the source trajectory $\mathcal{C} = \{v(t), t \in T\}$, is outside of Ω which is the convex hull of the support of μ, i.e., $\Omega \cap \mathcal{C} = \emptyset$. $\mathcal{D}_{\vec{v}}\mu\left(\vec{\zeta}\right) \stackrel{\text{def}}{=} \mathcal{D}\mu\left(\vec{v}, \vec{\zeta}\right)$ are called projections. In practice, the source trajectory is sampled. The number $p \in \mathbb{N}$ of x-ray projections is bounded.

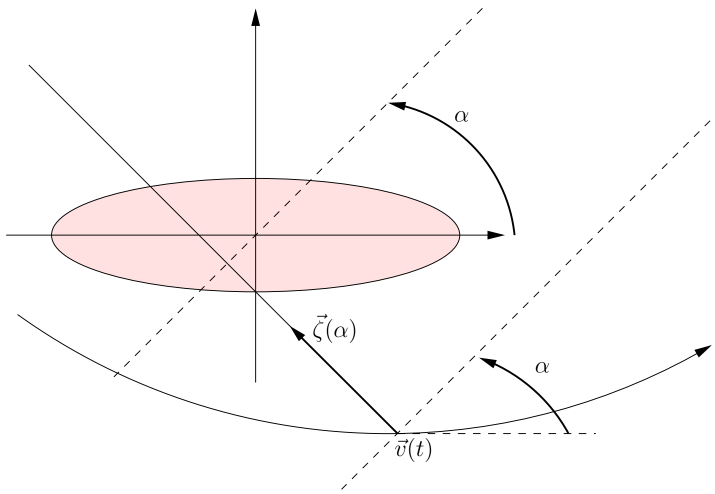

Figure 1: The Fan Beam variables (t, α), where $\vec{\zeta}(\alpha) = (-\sin\alpha, \cos\alpha)^t$. The cross section of the measured patient is supposed to be contained here in the ellipse.

Thus we deal with a finite number of vertexes, $\vec{v}_i \in \mathbb{R}^n, i = 1, \ldots, p$ (and $\vec{v}_i = \vec{v}(t_i), t_i \in T$ is the sampling of the source trajectory). In 2D CT, the well-known Filtered Back Projection formula yields an efficient inversion, i.e., the stable analytic reconstruction of μ from (1) when $\mathcal{D}\mu\left(\vec{v}, \vec{\zeta}\right)$ is acquired on a circular trajectory surrounding the measured object μ, for all direction $\vec{\zeta} \in \mathbb{S}^1$ at each $\vec{v}(t_i) \in \mathcal{C}$, see [9].

A trans-axial projection truncation occurs at a given $\vec{v}(t_i)$ if $\mathcal{D}_{\vec{v}}\mu\left(\vec{\zeta}\right)$ is not measured for all lines $\vec{v} + \mathbb{R}\vec{\zeta}$ intersecting the support of μ. In recent years, Region Of Interest (ROI) methods have been proposed to reconstruct μ on ROI under conditions of the ROI and the set measured lines. Practical examples of trajectories involving small detectors relative to the size of the support of μ (and thus for which trans-axial truncation can not be avoided, see Fig. 2) have been proposed for ROI reconstruction. A very good review of 2D ROI reconstruction approaches has been presented in [2]. 3D CB developments also exist.

In dynamic tomography or 3D reconstruction, we can no longer suppose that the function μ is not changing during the acquisition. This problem arises for example when measuring X-ray projections from the thorax region with a relative slow acquisition system like a C-arm. Let t the source trajectory parameter represent time, then μ is both a function of t and the spacial variable $\vec{x}$, $\mu(t, \vec{x})$. When the variations of μ during the acquisition is occurring just because of movements or time dependent space deformations, the assumption that $\mu(t, \vec{x})$ behaves like $\mu\left(\vec{\Gamma}_t\left(\vec{x}\right)\right)$ can be made, where μ

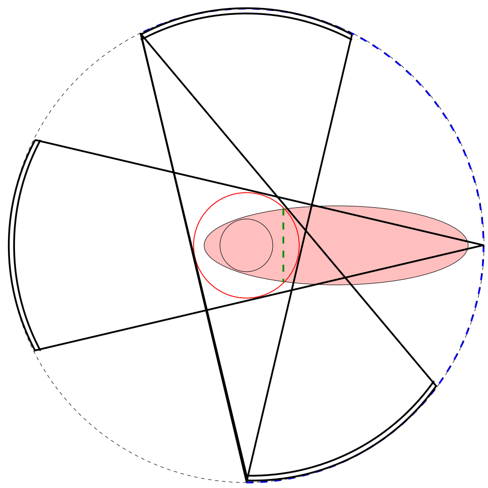

Figure 2: Small detector yields truncated data.

is the attenuation function at a reference time, for example $t = 0$, (in this case $\vec{\Gamma}_0(\vec{x}) = \vec{x}$) and $\vec{\Gamma}_t$ is a time dependent diffeomorphic[1] deformation, i.e. a smooth bijective mapping on the space $\mathbb{R}^n$:

$$\begin{array}{rccc} \vec{\Gamma}_t : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & \vec{x} & \longrightarrow & \vec{\Gamma}_t(\vec{x}) \end{array} . \tag{2}$$

Thus $\vec{\Gamma}_t(\vec{x})$ maps $\vec{x}$ at time t to its position at the reference time. This kind of modeling was introduced by Crawford et al [6] and further studied by Roux et al [12].

In divergent geometry, we define

$$\mathcal{D}\mu_{\vec{\Gamma}_t}(\vec{v}(t), \vec{\zeta}) = \int_{\mathbb{R}} \mu \left(\vec{\Gamma}_t \left(\vec{v}(t) + l\vec{\zeta} \right) \right) \left| \det J_{\vec{\Gamma}_t}(\vec{y} + l\vec{\zeta}) \right| dl \tag{3}$$

If we assume that $\vec{\Gamma}_t$ is known then μ has to be reconstructed from Eq. (3). We proposed in [8] a generalization of the analytic deformation compensation to the class of deformations preserving the acquisition line geometry with the restriction of linear deformation along each line. This last restriction was suppressed in [7] for deformations with mass conservation. Moreover, the compensation was extended in 2D to ROI reconstructions.

The estimation of the organ movements $\vec{\Gamma}_t$ is a very difficult and important question in practice. We have recently provided some solutions involving the Data Consitency Conditions (DCC), see [1, 5, 3, 10, 11, 4]

[1]If $\vec{\Gamma}_t$ and $\vec{\Gamma}_t^{-1}$ are r times continuously differentiable, $\vec{\Gamma}_t$ is called a C^r-diffeomorphism. We will suppose that $\vec{\Gamma}_t$ is at least a C^1-diffeomorphism

PhD research project The aim of the post-doc project is to improve Dynamic CB CT reconstruction. We want to extend our understanding of ROI reconstruction and dynamic Cone Beam CT. We first want to combine ROI and Dynamic reconstruction based on known results in each domain. A second direction of research would be to consider more complex deformations $\vec{\Gamma}_t$ such as deformations transforming the measured lines into circles or ellipses. Reconstruction of functions from integrals on circles or spheres is an active research field. A third direction would be to be to separate a moving organ (typically cancer calcification in lungs) within a fixed attenuation function and, knowing the deformation, to reconstruct the moving organ and the fixed one. A fourth direction of research would be to identify both the movement $\vec{\Gamma}_t$ and the function μ from the data. Generalizations within the framework of 2D ROI and 3D CB ROI reconstruction should also be considered. Numerical experiments and validation on real data (acquired locally or at Centre Léon Bérard in Lyon) will be necessary.

Acknowledgment

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