ANR PhD fellowship
Mathematics and Numerics of Dynamic Cone Beam CT and ROI Reconstructions
A mathematical introduction

Laurent Desbat
TIMC-IMAG, UMR 5525,
Grenoble University, France
Laurent.Desbat@imag.fr

Rolf Clackdoyle
LHC, UMR 5516,
Saint Etienne University, France
rolf.clackdoyle@univ-st-etienne.fr

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Let $\vec{v} \in \mathbb{R}^n$ denote a point (an X-ray source position in X-ray CT $n = 2$ or CB CT $n = 3$), and $\vec{\zeta} \in \mathbb{S}^{n-1}$ a unit vector (in the direction from the source to the detector in X-ray CT), the Divergent Beam transform is defined by

$$\mathcal{D}\mu\left(\vec{v}, \vec{\zeta}\right) = \int_0^{+\infty} \mu\left(\vec{v} + l\vec{\zeta}\right) dl \tag{1}$$

Generally (in particular in X-ray CT, see Fig. 1 for the Fan Beam geometry) the data are acquired from multiple source positions and the source follows a trajectory along a curve

$$\begin{array}{rcl} \vec{v}\ :\ T \subset \mathbb{R} & \longrightarrow & \mathbb{R}^n \\ t & \longrightarrow & \vec{v}(t) \end{array}$$

We will suppose in the following that the source trajectory $\mathcal{C} = \{v(t), t \in T\}$, is outside of Ω which is the convex hull of the support of μ, i.e., $\Omega \cap \mathcal{C} = \emptyset$. $\mathcal{D}_{\vec{v}}\mu\left(\vec{\zeta}\right) \stackrel{\text{def}}{=} \mathcal{D}\mu\left(\vec{v}, \vec{\zeta}\right)$ are called projections. In practice, the source trajectory is sampled. The number $p \in \mathbb{N}$ of x-ray projections is bounded. Thus we deal with a finite number of vertexes, $\vec{v}_i \in \mathbb{R}^n, i = 1, \ldots, p$ (and $\vec{v}_i = \vec{v}(t_i), t_i \in T$ is the sampling of the source trajectory). In 2D CT, the well-known Filtered Back Projection formula yields an efficient inversion, i.e., the stable analytic reconstruction of μ from (1) when $\mathcal{D}\mu\left(\vec{v}, \vec{\zeta}\right)$ is acquired on a circular trajectory surrounding the measured object μ, for all direction $\vec{\zeta} \in \mathbb{S}^1$ at each $\vec{v}(t_i) \in \mathcal{C}$, see [5].

A trans-axial projection truncation occurs at a given $\vec{v}(t_i)$ if $\mathcal{D}_{\vec{v}}\mu\left(\vec{\zeta}\right)$ is not measured for all lines $\vec{v} + \mathbb{R}\vec{\zeta}$ intersecting the support of μ. In recent

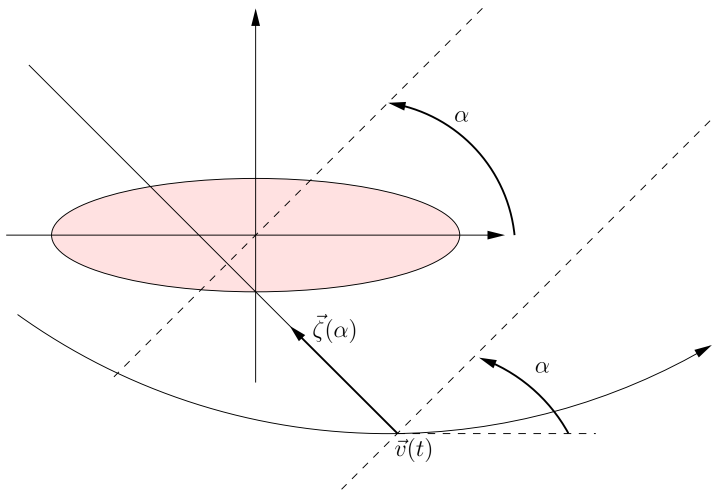

Figure 1: The Fan Beam variables (t, α), where $\vec{\zeta}(\alpha) = (-\sin\alpha, \cos\alpha)^t$. The cross section of the measured patient is supposed to be contained here in the ellipse.

years, Region Of Interest (ROI) methods have been proposed to reconstruct μ on ROI under conditions of the ROI and the set measured lines. Practical examples of trajectories involving small detectors relative to the size of the support of μ (and thus for which trans-axial truncation can not be avoided, see Fig. 2) have been proposed for ROI reconstruction. A very good review of 2D ROI reconstruction approaches has been presented in [1]. 3D CB developments also exist.

In dynamic tomography or 3D reconstruction, we can no longer suppose that the function μ is not changing during the acquisition. This problem arises for example when measuring X-ray projections from the thorax region with a relative slow acquisition system like a C-arm. Let t the source trajectory parameter represent time, then μ is both a function of t and the spacial variable $\vec{x}$, $\mu(t, \vec{x})$. When the variations of μ during the acquisition is occurring just because of movements or time dependent space deformations, the assumption that $\mu(t, \vec{x})$ behaves like $\mu\left(\vec{\Gamma}_t\left(\vec{x}\right)\right)$ can be made, where μ is the attenuation function at a reference time, for example $t = 0$, (in this case $\vec{\Gamma}_0(\vec{x}) = \vec{x}$) and $\vec{\Gamma}_t$ is a time dependent diffeomorphic[1] deformation, i.e. a smooth bijective mapping on the space $\mathbb{R}^n$:

$$\begin{array}{rccc} \vec{\Gamma}_t : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & \vec{x} & \longrightarrow & \vec{\Gamma}_t(\vec{x}) \end{array} . \tag{2}$$

[1]If $\vec{\Gamma}_t$ and $\vec{\Gamma}_t^{-1}$ are r times continuously differentiable, $\vec{\Gamma}_t$ is called a C^r-diffeomorphism. We will suppose that $\vec{\Gamma}_t$ is at least a C^1-diffeomorphism

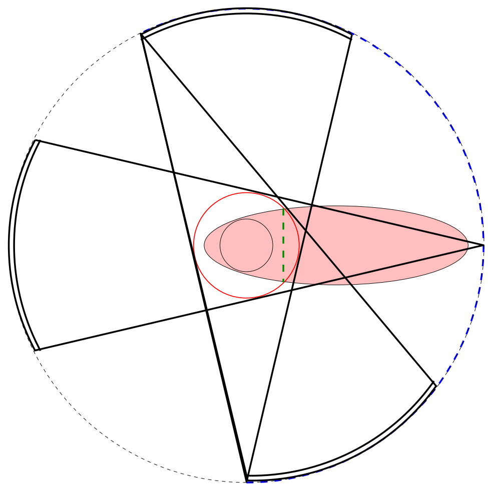

Figure 2: Small detector yields truncated data.

Thus $\vec{\Gamma}_t(\vec{x})$ maps $\vec{x}$ at time t to its position at the reference time. This kind of modeling was introduced by Crawford et al [2] and further studied by Roux et al [6].

In divergent geometry, we define

$$\mathcal{D}\mu_{\vec{\Gamma}_t}(\vec{v}(t), \vec{\zeta}) = \int_{\mathbb{R}} \mu\left(\vec{\Gamma}_t\left(\vec{v}(t) + l\vec{\zeta}\right)\right)\left|\det J_{\vec{\Gamma}_t}(\vec{y} + l\vec{\zeta})\right| dl \tag{3}$$

If we assume that $\vec{\Gamma}_t$ is known then μ has to be reconstructed from Eq. (3). We proposed in [4] a generalization of the analytic deformation compensation to the class of deformations preserving the acquisition line geometry with the restriction of linear deformation along each line. This last restriction was suppressed in [3] for deformations with mass conservation. Moreover, the compensation was extended in 2D to ROI reconstructions.

References

[1] R. Clackdoyle and M. Defrise. Tomographic reconstruction in the 21st century. *IEEE Signal Processing Magazine*, 27(4):60–80, 2010.

[2] C.R. Crawford, K.F. King, C.J. Ritchie, and J.D. Godwin. Respiratory compensation in projection imaging using a magnification and displacement model. *IEEE Transactions on Medical Imaging*, 15:327 –332, 1996.

[3] L. Desbat, C. Mennessier, and R. Clackdoyle. Dynamic tomography, mass preservation and ROI reconstruction. In *CT 2012, Salt Lake City*, pages 318–322, 2012.

[4] L. Desbat, S. Roux, and P. Grangeat. Compensation of some time dependent deformations in tomography. *IEEE transaction on Medical Imaging*, 26(2):261–269, 2007.

[5] F. Natterer. *The Mathematics of Computerized Tomography.* Wiley, 1986.

[6] S. Roux, L. Desbat, A. Koenig, and P. Grangeat. Exact reconstruction in 2D dynamic CT: compensation of time-dependent affine deformations. *Phys. Med. Biol.*, 49(11):2169–82, June 2004.